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ON THE COMPUTATION OF STABILIZED TENSOR FUNCTORS AND THE RELATIVE ALGEBRAIC K-THEORY OF DUAL NUMBERS

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ABSTRACT. We compute the stabilization of functors from exact categories to abelian groups derived from n-fold tensor products. Rationally, this gives a new computation for the relative algebraic K-theory of dual numbers.

INTRODUCTION

In [8], T. Goodwillie computed the relative algebraic K-theory of dual numbers rationally and used this to show that a suitable trace map to cyclic homology is a rational equivalence on relative theories for nilpotent extensions. In [5], a new model for the relative algebraic K-theory of dual numbers was introduced and used to show the equivalence of stable K-theory and topological Hochschild homology—which was shown in [16] to be equivalent to Mac Lane homology [10]. One of our goals in this paper is to give a new computation of Goodwillie's result by exploiting the model from [5]. To do this, one is lead to compute the *stabilization* of various functors from exact categories to abelian groups, as introduced in [14]. Essentially, given a functor F from exact categories to abelian groups, its stabilization is defined as: $F^{st} = \lim_{n\to\infty} F(S^{(n)}\ )[-n]$, where $S^{(n)}$ is Waldhausen's S construction for exact categories ([17]) iterated n times. This can be thought of as a generalization of the bar construction for abelian groups, and hence this stabilization is a direct transliteration of the Dold-Puppe stable derived functors ([4]) to the setting of exact categories. Our method of computation for the stabilized functors is first to relate them to a stabilized version of the cohomology of small categories in the sense of [2] and then to relate this to Mac Lane homology much in the manner of [9].

There is a functor $\mathcal{S}_*$ from exact categories to categories of exact categories such that $\mathcal{S}_{\mathcal{A}}$ is the smallest subcategory of all exact categories containing $\mathcal{A}$ which is closed under isomorphisms and taking Waldhausen's $S.$ construction (see section 0 for more details). Let R be a ring and let M be an R–bimodule. Let $\mathcal{M}$ be the category of all (right) R–modules and let $\mathcal{P}$ be the full category of finitely generated projective R–modules. We write $\mathcal{S}_I$ for $\mathcal{S}_*$ of the exact inclusion functor $I : \mathcal{P} \longrightarrow \mathcal{M}$, and similarly $\mathcal{S}_M$ for $\mathcal{S}_*$ of the exact functor $\star \otimes_R M$ from $\mathcal{P}$ to $\mathcal{M}$. Let $Q_*(R)$ be Mac Lane's Q–construction ([10]).

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Theorem (4.1). *Let G be the functor from $\mathcal{S}_{\mathcal{P}}$ to abelian groups defined by*

$$G(\mathcal{A}) = \bigoplus_{A\in\mathcal{A}} \bigotimes_{i=1}^{n} Hom_{\mathcal{S}_I(\mathcal{A})}(\mathcal{S}_I(A), \mathcal{S}_M(A)).$$

Then G^{st} is completely determined by its value at $\mathcal{P}$, and

$$G^{st}(\mathcal{P}) \sim_{\Sigma_n} \mathbf{Z}[\Sigma_n] \otimes_{\mathbf{Z}[C_n]} HH\left(Q_*(R)^{\otimes n}; M_{\tau}^{\otimes n}\right),$$

where HH is the Hochschild homology complex for $Q_*(R)^{\otimes n}$ acting on the bimodule by

$$(m_1 \otimes \cdots \otimes m_n) * (q_1 \otimes \cdots \otimes q_n) = (m_1q_1 \otimes m_2q_2 \otimes \cdots \otimes m_nq_n),$$

$$(q_1 \otimes \cdots \otimes q_n) * (m_1 \otimes \cdots \otimes m_n) = (q_nm_1 \otimes q_1m_2 \otimes \cdots \otimes q_{n-1}m_n),$$

and C_n is the cyclic group of n elements which acts by cyclic permutations—the equivalence is weakly Σ_n–equivariant.

The paper is organized as follows. In section 0 we recall some terminology from [17] and establish some notation. In section 1 we show how to reduce the computation of relative algebraic K-theory of dual numbers rationally to that of the stabilization of functors like G in the above proposition. In section 2 we generalize a result of [7] to rewrite these in terms of a suitably stabilized cohomology of small categories. In section 3 we further reduce these models to appropriate (no longer stabilized) cohomology of small categories. In section 4 we reinterpret these results in terms of Mac Lane homology following the ideas of [9].

0. Preliminaries—making a functor additive by stabilization

In this section we recall the definition of the S construction from [17] and establish some notation. We then recall the notion of *stabilization* for functors from exact categories to chain complexes as introduced in [14].

For $q \in \mathbf{N}$, let $[q]$ denote the poset $\{0 < \cdots < q\}$, which we will often view as a category. For $\mathcal{C}$ (small) and $\mathcal{D}$ categories, let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the category of functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms the natural transformations of these. For $\mathcal{D}$ a category, let the *arrow category*, $\mathrm{Ar}\,\mathcal{D}$, be $\mathrm{Fun}([1], \mathcal{D})$.

Let $\mathcal{E}$ be an exact category considered as a category with cofibrations ([17], first few lines) by setting the subcategory of cofibrations to be the admissible monomorphisms. We let $\mathrm{Ex}(\mathrm{Ar}[q], \mathcal{E})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[q], \mathcal{E})$ whose objects are the functors F such that $F(j \to j) = *$ and, for every triple $i \le j \le k$ in $[q]$,

$$F(i \to j) \to F(i \to k) \to F(j \to k)$$

is a short exact sequence. Setting $S_{[q]}\mathcal{E} = \mathrm{Ex}(\mathrm{Ar}[q], \mathcal{E})$,we obtain a simplicial exact category, and we write $S\mathcal{E}$ for both this simplicial exact category and the associated simplicial set we obtain by taking the set of objects degreewise. We can iterate the S construction, and we write $\mathbf{K}\mathcal{C}$ for the algebraic K-theory (pre-)spectrum of $\mathcal{C}$, $\mathbf{K}\mathcal{C} = \{|S^{(n)}\mathcal{C}|\}_{n\ge 0}$ (with structure maps constructed by the natural isomorphism $\mathcal{C} \cong S_1\mathcal{C}$).

Conventions: We will make no notational distinction between a (multi-dimensional) simplicial abelian group and its associated (multi-dimensional) chain complex. By a *chain complex* we will always mean a complex which is bounded below and homologically trivial in negative dimensions (i.e. connective). Given a multi-dimensional chain complex, we will consider it as a chain complex by taking Tot (using products). What follows has standard generalizations to various categories with cofibrations; but as these extensions are "straightforward" for the expert and do little more than cloud the essential ideas, we will keep our attention to exact categories.

Let F be any functor from (small) linear categories (with a distinguished zero object 0) to chain complexes. It will be convenient for us to assume further that F is *reduced*. That is, that $F(0) = 0$. We will always think of an abelian group as a chain complex concentrated in dimension 0.

We note that we are not assuming that F takes naturally equivalent linear categories to homotopic chain complexes. If F does this, we will say that F is an *equivalence functor*. If $G.$ is any functor from some category $\mathcal{C}$ of categories to (small) simplicial linear categories, then we can of course compose functors to obtain a new functor $FG.$ from $\mathcal{C}$ to simplicial chain complexes, which we once again consider as a functor to chain complexes by taking Tot. By definition, if $\mathcal{A}_*$ is a simplicial (small) linear category, then $F\mathcal{A}_*$ is the simplicial chain complex obtained by applying F degreewise.

We will say that F is *product preserving* if for any two (small) exact categories $\mathcal{A}$ and $\mathcal{B}$, the natural projection map ρ of simplicial abelian groups $F(\mathcal{A} \times \mathcal{B})$ to $F(\mathcal{A}) \times F(\mathcal{B})$ is a homotopy equivalence. We will say that F is a p-product functor if F preserves products in a range $0 \le i \le p$ (that is, $\pi_i(\rho)$ is an isomorphism for all $0 \le i \le p$). By the proof of additivity found in [12], for any F

$$FS.S_2\mathcal{C} \xrightarrow{d_0 \times d_2} FS.(\mathcal{C} \times \mathcal{C})$$

is an equivalence. If F is a p–product functor, then the natural map $F(SS_2) \xrightarrow{d_0 \times d_2} FS \times FS$ is an equivalence in a p–range. If $FS_2 \longrightarrow F \times F$ is an equivalence in a p range, then we say that F is *additive* in a p range.

Lemma (1.5 of [14]). *For any $n \ge 1$, the functor $FS^{(n)}$ is a reduced equivalence functor which is a $2n-1$ product functor and additive in a $2n-1$ range.*

For X a chain complex, we let $X[z]$ be the new chain complex with $X[z]_n = X_{n-z}$ and $\partial[z]_n = \partial_{n-z}$.

Definition (0.1). For any exact category, we define

$$F_*^{st}(\mathcal{A}) = \lim_{n \to \infty} FS^{(n)}\mathcal{A}[-n],$$

which is a natural additive equivalence functor. We let $\alpha : F \to F^{st}$ be the natural transformation obtained by the structure maps for the limit system. By lemma 1.7 of [14], F is an additive functor (additive in an ∞-range) if and only if $F \to F^{st}$ is an equivalence for all exact categories.

Examples. 1) Let $\mathbf{Z}$ be the functor which takes a (small) category $\mathcal{C}$ to the reduced free abelian group generated by the set of objects of $\mathcal{C}$. That is,

$$\mathbf{Z}(\mathcal{C}) = cokernel[\mathbf{Z}[0] \to \mathbf{Z}[Obj(\mathcal{C})]].$$

Then, by [14], $\mathbf{Z}^{st}$ is the stable homology functor and

$$H_*(\mathbf{Z}^{st}(\mathcal{C})) = H_*(\mathbf{K}(\mathcal{C})) = \pi_*(\mathbf{K}(\mathcal{C}) \wedge \mathbf{HZ}).$$

2) Let $\mathcal{A}$ be an exact category. Let **Hom** be the functor defined by

$$\mathbf{Hom}(\mathcal{A}) = \bigoplus_{A \in \mathcal{A}} Hom_{\mathcal{A}}(A, A).$$

Then, by [6],

$$\mathbf{Hom}^{st}(\mathcal{A}) = THH(\mathcal{A}),$$

where THH is the topological Hochschild homology of $\mathcal{A}$.

If F is a functor defined on a subcategory $\mathcal{S}$ of all exact categories, then in order for F^{st} to still be defined we simply need that if $\mathcal{A} \in \mathcal{S}$ then $S.\mathcal{A}$ is a simplicial $\mathcal{S}$–object. In what follows we will need to restrict ourselves to functors defined on such subcategories of all exact categories. In particular, if $\mathcal{A}$ is an exact category we let $\mathcal{S}_{\mathcal{A}}$ be the smallest subcategory which contains $\mathcal{A}$, is closed under taking $S.\mathcal{A}$ (that is, $S.\mathcal{A}$ is a simplicial $\mathcal{S}_{\mathcal{A}}$ object) and is closed under isomorphisms (if an exact category $\mathcal{E}$ is isomorphic to an exact category in $\mathcal{S}_{\mathcal{A}}$ then it is in $\mathcal{S}_{\mathcal{A}}$). The category $\mathcal{S}_{\mathcal{A}}$ is skeletally small and is equivalent to the category with objects $S_{n_1} S_{n_2} \cdots S_{n_t}\mathcal{A}$ for all $t \geq 0$ and finite sequences $(n_1, \ldots, n_t)$ of non-negative integers with morphisms those determined by the $S.$ construction. In this way we see that for any exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ we get a functor $\mathcal{S}_F : \mathcal{S}_{\mathcal{A}} \longrightarrow \mathcal{S}_{\mathcal{B}}$ determined by $S_{n_1} S_{n_2} \cdots S_{n_t} F$ for each finite sequence $(n_1, \ldots, n_t)$ of non-negative integers. Thus, $\mathcal{S}_*$ is a functor from the category of exact categories to (skeletally small) categories of exact categories. For F an exact functor from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{E} \in \mathcal{S}_{\mathcal{A}}$ and E an object of $\mathcal{E}$, we abuse notation as follows: $\mathcal{S}_F$ is a functor from $\mathcal{S}_{\mathcal{A}}$ to $\mathcal{S}_{\mathcal{B}}$ which produces a functor $\mathcal{S}_F|_{\mathcal{E}}$ from $\mathcal{E}$ to $S_F(\mathcal{E})$, and we set

$$S_F(E) = S_F|_{\mathcal{E}}(E) \in S_F(\mathcal{E}).$$

Example. 3) Let R be a ring and let M be an R–bimodule. Let $\mathcal{M}$ be the category of all (right) R–modules and let $\mathcal{P}$ be the full category of finitely generated projective R–modules. Let I be the exact inclusion functor $I : \mathcal{P} \longrightarrow \mathcal{M}$ and let M be the exact functor $\star \otimes_R M$ from $\mathcal{P}$ to $\mathcal{M}$. Let **M** be the functor from $\mathcal{S}_{\mathcal{P}}$ to abelian groups defined by

$$\mathbf{M}(\mathcal{A}) = \bigoplus_{A \in \mathcal{A}} Hom_{\mathcal{S}_I(\mathcal{A})}(\mathcal{S}_I(A), \mathcal{S}_M(A)).$$

By section 2 of [5],

$$\mathbf{M}^{st}(\mathcal{A}) = THH(R; M),$$

where $THH(R, M)$ is the topological Hochschild homology of the ring spectrum **HR** with coefficients in the bimodule **HM**.

1. On the computation of $\tilde{K}(R \oplus M)_{\mathbf{Q}}$

Definition. Following [5], for R a ring, M an R–bimodule and X a space (= finite pointed simplicial set) we define $\tilde{\mathbf{K}}(R, \tilde{M}[X])$ to be the connective (pre-)spectrum:

$$\tilde{K}(R, \tilde{M}[X])(n) = \left| [p] \times [q] \mapsto \bigvee_{\bar{P} \in S_q^{(n)}\mathcal{P}} Hom_{S_q^{(n)}\mathcal{M}}(\bar{P}, \bar{P} \otimes_R \tilde{M}[X_p]) \right|,$$

where $\mathcal{P}$ is the exact category of finitely generated R-modules, $\mathcal{M}$ the exact category of all R-modules and $\tilde{M}[X_p] = \bigoplus_{X_p - \text{basept}} M$. By section 4 of [5], if we let $R \oplus M$

be the ring with multiplication defined by $(r,m)(r',m') = (rr', rm' + mr')$, then $\mathbf{K}(R \oplus M)$ is naturally equivalent to $\mathbf{K}(R) \times \tilde{\mathbf{K}}(R; \tilde{M}[S^1])$.

For any connective spectrum (of CW-type), the Hurewicz map produces an isomorphism from the rational homotopy groups of the spectrum to its rational homology groups (see for example page 203 of [1]). Thus,

$$\begin{aligned}&\pi_n(\tilde{\mathbf{K}}(R \oplus M)) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \pi_n \tilde{\mathbf{K}}(R; \tilde{M}[S^1]) \otimes_{\mathbf{Z}} \mathbf{Q} \\ &\quad \xrightarrow{\cong} \lim_{k\to\infty} H_{n+k}(|[p] \times [q] \mapsto \bigvee_{\bar{P} \in S_q^{(k)}\mathcal{P}} Hom_{S_q^{(k)}\mathcal{M}}(\bar{P}, \bar{P} \otimes_R \tilde{M}[X_p])|; \mathbf{Q}).\end{aligned}$$

In general, for any (simplicial) bimodule M and abelian group G, we obtain

$$\begin{aligned}H_n(\tilde{\mathbf{K}}(R;M);G) &\simeq \lim_{k\to\infty} \pi_{n+k}|[q] \mapsto \tilde{G}\left[\bigvee_{\bar{P} \in S_q^{(k)}\mathcal{P}} Hom_{S_q^{(k)}\mathcal{M}}(\bar{P}, \bar{P} \otimes_R M)\right]| \\ &\simeq \lim_{k\to\infty} \pi_{n+k}|[q] \mapsto \bigoplus_{\bar{P} \in S_q^{(k)}\mathcal{P}} \tilde{G}\left[Hom_{S_q^{(k)}\mathcal{M}}(\bar{P}, \bar{P} \otimes_R M)\right]| \\ &= \tilde{G}[M]^{st}(\mathcal{P}),\end{aligned}$$

where $\tilde{G}[M]$ is the functor from $\mathcal{S}_{\mathcal{P}}$ to simplicial abelian groups defined by

$$\tilde{G}[M](\mathcal{A}) = \bigoplus_{A \in \mathcal{A}} \tilde{G}[Hom_{\mathcal{S}_I(\mathcal{A})}(\mathcal{S}_I(A), \mathcal{S}_M(A))]$$

and $G[M]^{st}$ is defined as in 0.1. Putting these remarks together, we obtain the following proposition.

Proposition (1.1). *For any ring R and R–bimodule M,*

$$\pi_n(\tilde{\mathbf{K}}(R \oplus M)) \otimes_{\mathbf{Z}} \mathbf{Q} \cong H_n(\tilde{\mathbf{Q}}[B.M]^{st}(\mathcal{P})),$$

where $B.M = \tilde{M}[S^1]$ is the usual bar construction for the abelian group M considered as a simplicial R–bimodule.

To examine $\tilde{\mathbf{Q}}[B.M]$, we first recall some well known results. If G is an abelian group, we let $p^n : G \longrightarrow G^{\otimes n}$ be the map of pointed *sets* defined by

$$p^n(g) = \overbrace{g \otimes \cdots \otimes g}^{n \text{ times}}.$$

We abuse notation, and also write p^n for the composed map

$$G \xrightarrow{p^n} G^{\otimes n} \xrightarrow{\rho} G_{\mathbf{Q}} \otimes \cdots \otimes G_{\mathbf{Q}} \otimes_{\mathbf{Q}[\Sigma_n]} \mathbf{Q} = S^n(G_{\mathbf{Q}}),$$

where $G_{\mathbf{Q}} = G \otimes_{\mathbf{Z}} \mathbf{Q}$, ρ is the natural map and Σ_n acts on the tensor product by permuting factors. Extending by linearity, we obtain a natural transformation of functors from abelian groups to rational vector spaces

$$\tilde{\mathbf{Q}}[G] \xrightarrow{p} \prod_{n \in \mathbf{N}} S^n(G_{\mathbf{Q}}).$$

We extend p to simplicial abelian groups by evaluating everything degreewise.

In general, the map p is not an isomorphism, but it is an equivalence for 0–connected simplicial abelian groups. One can see this as follows. First we recall that (see for example theorem V.7.6 of [18])

$$H_*(K(\mathbf{Z}/m\mathbf{Z};1);\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i=0, \\ \mathbf{Q}, & i=1 \text{ and } m=0, \\ 0, & \text{otherwise.} \end{cases}$$

By the Künneth theorem, this implies that

$$H_n(K(G;1),\mathbf{Q}) = (G_{\mathbf{Q}} \otimes \cdots \otimes G_{\mathbf{Q}})^{sgn} \otimes_{\mathbf{Q}[\Sigma_n]} \mathbf{Q} = \bigwedge^n (G_{\mathbf{Q}}),$$

where $(\ )^{sgn}$ indicates we are now taking the Σ_n action with signs and so $\bigwedge^n$ is the n-th exterior power. Now, $S^n(B.G_{\mathbf{Q}})$ is simply $B^n G_{\mathbf{Q}}^{\otimes n}/\Sigma_n$, which (because the action of Σ_n on the deloopings gives a signed action on the homotopy groups) is simply $B^n((G_{\mathbf{Q}}^{\otimes n})^{sgn}/\Sigma_n)$ (we are over $\mathbf{Q}$), and hence the result (after checking that the given transformation does indeed provide the correct map).

Corollary (1.2). *Putting together the above remarks, we see that*

$$(\tilde{\mathbf{Q}}[B.M])^{st} \xrightarrow{\simeq} \bigoplus_{n=1}^{\infty} (S^n[B.M_{\mathbf{Q}}])^{st}.$$

Next, we recall that since we are working over the rationals, a Σ_n equivariant map which is also a weak equivalence is a weak equivalence on the map of orbits (since $|\Sigma_n| = n!$ is invertible). Let $T^n(M)$ be the functor from $\mathcal{S}_{\mathcal{P}}$ to $\mathbf{Z}[\Sigma_n]$–modules defined by

$$T^n(M)(\mathcal{A}) = \bigoplus_{A\in\mathcal{A}} \bigotimes_{i=1}^{n} Hom_{\mathcal{S}_I(\mathcal{A})}(\mathcal{S}_I(A), \mathcal{S}_M(A)).$$

The following is a special case of the more general result in (4.1):

Proposition (1.3). *Let* $\mathbf{Q} \subseteq R$. *Then*

$$T^n(B.M)^{st}(\mathcal{P}) \sim_{\mathbf{Q}[\Sigma_n]} \mathbf{Q}[\Sigma_n] \otimes_{\mathbf{Q}[C_n]} HH(R^{\otimes n}; B.M_\tau^{\otimes n}),$$

where HH *is the Hochschild homology and* τ *indicates the cyclic twisted action of* $R^{\otimes n}$ *on* $X^{\otimes n}$ *from the introduction. The cyclic group of order* n, C_n, *acts by permuting tensor factors. Hence*

$$S^n(B.M)^{st}(\mathcal{P}) \simeq HH(R^{\otimes n}; B.M_\tau^{\otimes n})]/C_n$$

and

$$\frac{K(R\oplus M)}{K(R)} \simeq_{\mathbf{Q}} \bigoplus_{n=1}^{\infty} HH(R^{\otimes n}; B.M_\tau^{\otimes n})]/C_n.$$

Remark. Using the techniques of [13] and the explicit maps used to obtain the above result, it is straightforward to show that the trace map from algebraic K-theory to negative homology used in [8] produces the needed isomorphism on relative theories after tensoring with the rationals. Since our objective here is to study the stabilized tensor functors, we will only give a brief outline below of how this can be done, and leave further details to the interested reader.

Aside (1.4). On the rational equivalence of relative algebraic K-theory and relative negative cyclic homology.

The natural ring map $R \longrightarrow R \oplus M$ (taking r to $(r, 0)$) produces a natural exact functor $\epsilon_M : \mathcal{P}_R \longrightarrow \mathcal{P}_{R\oplus M}$. By page 218 of [15] we have a natural transformation Φ of functors from $\mathcal{S}_P \times R\text{-}Mod\text{-}R$ to cyclic $\mathbf{Q}$-modules

$$\Phi : \tilde{\mathbf{Q}}[N^{cy}*](\star) \longrightarrow HH(\mathcal{S}_{\epsilon_*}\star).$$

One always has a natural simplicial map ρ from $B.M$ to $N^{cy}M$ defined by sending $(m_1, \dots, m_n)$ to $(-(\Sigma_i m_i), m_1, \dots, m_n)$, and the following diagram commutes (section 4 of [15]):

$$\begin{array}{ccc}
\mathbf{K}(R\oplus M) & \xrightarrow{trace} & HH(R\oplus M) \\
\big\downarrow & & \big\uparrow \Phi \\
\tilde{\mathbf{Q}}[B.M]^{st}(\mathcal{P}) & \xrightarrow{\rho} & \tilde{\mathbf{Q}}[N^{cy}M]^{st}(\mathcal{P})
\end{array}$$

(the left vertical map is the composite of the equivalence from [5] with the Hurewicz map). In general, one can decompose $HH(R\oplus M)$ as the direct sum of cyclic abelian groups $\bigoplus_{i=0}^{\infty} HH^{[i]}(R|M)$, where $HH^{[i]}(R|M)_{[p]}$ is the submodule of $HH(R\oplus M)_{[p]}$ determined by sums of tensors $(x_0 \otimes \cdots \otimes x_n)$ with exactly i of the x_j's in M. One can extend this definition to $HN^{[i]}(\mathcal{S}_{\epsilon_*\star})$ as a functor from $\mathcal{S}_P$ to (unbounded) chain complexes which is once again additive. For each $n \geq 0$ we obtain a factorization (up to equivalence)

$$\begin{array}{ccc}
\tilde{\mathbf{Q}}[N^{cy}*](\star) & \xrightarrow{\Phi} & HH(\mathcal{S}_{\epsilon_*}(\star)) \\
\big\downarrow p^n & & \big\downarrow \pi \\
T^n(N^{cy}*)(\star) & \longrightarrow & HH^{[n]}(\mathcal{S}_{\epsilon_*}(\star)) \\
\big\uparrow inc & & \big\uparrow \\
F^n(N^{cy}*)(\star) & \longrightarrow & HN^{[n]}(\mathcal{S}_{\epsilon_*}(\star))
\end{array}$$

(p^n takes $[m]$ to $m \otimes \cdots \otimes m$). In the above diagram, $F^n = (\bigotimes^n\ )^{\Sigma_n}$, where the Σ_n action is given by permuting tensor factors and the map *inc* is given by the inclusion $F^n \longrightarrow T^n$. Since we are working over $\mathbf{Q}$, the norm map from S^n to F^n is an equivalence which corresponds to the map from cyclic homology (orbits) to negative homology (fixed points), being an equivalence in this situation. By looking carefully at the computation for $HN(R\oplus M)$ in [8], one sees that the composite map $F^n(B.*)(\star)^{st} \xrightarrow{\rho} HN^{[n]}(S_{\epsilon_*}\star)^{st} \xleftarrow{\simeq} HN^{[n]}(S_{\epsilon_*}\star)$ is a rational equivalence, and hence the lift of the trace map to negative cyclic homology is rationally a relative equivalence for split square zero ring extensions.

This ends aside 1.4.

2. A relation between stabilized functors and stabilized homology of small categories

In this section we slightly generalize a result from [7] which relates the stabilization (in the sense of 0.1) of a small class of functors to their appropriately stabilized Hochschild-Mitchell homology.

Definition. (See [2].) Let $\mathcal{A}$ be a small category and let $D : \mathcal{A}^{op} \times \mathcal{A} \to Ab$ be a bifunctor from $\mathcal{A}$ to abelian groups. We let $F_*(\mathcal{A}; D)$ be the simplicial abelian group defined by setting

$$F_p(\mathcal{A}; D) = \bigoplus_{\vec{A} \in N_p\mathcal{A}} D(A_1, A_0), \qquad \vec{A} = A_1 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_p} A_0.$$

If we represent an element of one component by $(g; \alpha_1, \dots, \alpha_p)$, then the face and degeneracy operators are given by

$$d_i(g; \alpha_1, \dots, \alpha_p) = \begin{cases} (D(\alpha_1, id)(g); \alpha_2, \dots, \alpha_p), & i = 0, \\ (g; \dots, \alpha_i\alpha_{i+1}, \dots \alpha_p), & 1 \le i \le p-1, \\ (D(id, \alpha_p)(g); \alpha_1, \dots, \alpha_{p-1}), & i = p, \end{cases}$$

$$s_i(g; \alpha_1, \dots, \alpha_p) = \begin{cases} (g; \dots, \alpha_i, id_{A_{i+1}}, \alpha_{i+1}, \dots), & 0 \le i \le p-1, \\ (g; \alpha_1, \dots, \alpha_p, id_{A_0}), & i = p. \end{cases}$$

The homology of $F_*(\mathcal{C}; D)$ is the *Hochschild-Mitchell* homology of the category $\mathcal{C}$ with coefficients in the bifunctor D.

Definition. Let $\mathcal{E}$ be an exact category. A *local coefficient system* G (at $\mathcal{E}$) associates a bifunctor $G_{\mathcal{A}}$ from $\mathcal{A}^{op} \times \mathcal{A}$ to simplicial abelian groups for each $\mathcal{A} \in \mathcal{S}_{\mathcal{E}}$ such that

(i) $G_{\mathcal{A}}$ is bireduced—$G(0, A) = 0 = G(A, 0)$ for all $A \in \mathcal{A}$
(ii) G is natural—for every morphism $F : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathcal{S}_{\mathcal{E}}$, there is a natural transformation of bifunctors $G_F : G_{\mathcal{A}} \longrightarrow G_{\mathcal{B}}$ such that $G_{id} = id$ and $G_{F \circ F'} = G_F \circ G_{F'}$.

Example. If $M_1, \dots, M_n$ are R–bimodules, we have a local coefficient system $G(M_1, \dots, M_n)$ at $\mathcal{P}$ given by

$$G_{\mathcal{A}}(A, A') = \bigotimes_{i=1}^{n} Hom_{\mathcal{S}_I(\mathcal{A})}(\mathcal{S}_I(A), \mathcal{S}_{M_i}(A')).$$

Notation. Let G be a local coefficient system for $\mathcal{E}$. By naturality, $F_*(\star, G_\star)$ is a functor from $\mathcal{S}_{\mathcal{E}}$ to simplicial abelian groups. For the purposes of proposition 2.1 below we will simply write $F_*(S^{(k)})$ for the $k+1$–simplicial abelian group determined by $F_*(S^{(k)}\mathcal{A}; G)$ when $\mathcal{A}$ and G are clear. Let δ be the natural transformation given by degeneracies from F_0 to F_*.

Proposition (2.1) (similar to [6] for the case $G(R)$)**.** *Let G be a local coefficient system for $\mathcal{E}$. The natural transformation $\delta(S^{(N)})$ from $F_0(S^{(N)})$ to $F_*(S^{(N)})$ is $2N-1$–connected, and hence δ^{st} (as in 0.1) is an equivalence.*

Proof. More generally, we show that for all $n \in \mathbf{N}$, the map from $F_0S^{(N)}\mathcal{A}$ to $F_nS^{(N)}\mathcal{A}$ given by degeneracies is $2N-1$–connected, which implies the result by a standard spectral sequence argument. Let c be the natural transformation from F_n to F_0 defined by sending $(g; \alpha_1, \dots, \alpha_n)$ to $(G(\alpha_1 \cdots \alpha_n; id)(g))$. Since $c \circ deg = id_{F_0}$, it suffices to show that $C = deg \circ c$ agrees with the identity in a $2N-1$ range when we include $S^{(N)}$ into the picture. In other words, we want to show that the simplicial self map C of $F_nS^{(N)}\mathcal{A}$ defined by sending $(g; \alpha_1, \dots, \alpha_n)$ to $(G(\alpha_1 \cdots \alpha_n; id)(g); id_{C_0}, \dots, id_{C_0})$ is $2N-1$ connected.

To prove this we are going to use the fact that $F_nS^{(N)}$ satisfies additivity in a $2N-1$ range. We construct three natural transformations T_{id}, T_{-c} and T_t

from F_n to F_nS_2, which then assemble to give simplicial maps from $F_nS^{(N)}\mathcal{A}$ to $F_nS^{(N)}S_2\mathcal{A}$. We define T_{id}, T_{-c} and T_t as follows.

Let $\vec{\alpha} = (g; C_1 \xleftarrow{\alpha_1} \cdots \longleftarrow C_n \xleftarrow{\alpha_n} C_0)$ be an element of $F_n(\mathcal{A}; G)$ and let $\alpha_{i\cdots j}$ be the composite $\alpha_i\alpha_{i+1}\cdots\alpha_j$. Then

$$T_{id}(\vec{\alpha}) = \left[\begin{array}{c|ccccccccc} & C_0 & = & C_0 & = & \dots & = & C_0 & \longleftarrow & 0 \\ & \downarrow{\scriptstyle i_{C_0}} & & \downarrow{\scriptstyle i_{C_0}} & & & & \downarrow{\scriptstyle i_{C_0}} & & \downarrow \\ G(1)(g); & C_0 \oplus C_1 & \xleftarrow{1\oplus\alpha_1} & C_0 \oplus C_2 & \xleftarrow{1\oplus\alpha_2} & \dots & \xleftarrow{1\oplus\alpha_{n-1}} & C_0 \oplus C_n & \xleftarrow{1,\alpha_n} & C_0 \\ & \downarrow{\scriptstyle \pi_{C_1}} & & \downarrow{\scriptstyle \pi_{C_2}} & & & & \downarrow{\scriptstyle \pi_{C_n}} & & \| \\ & C_1 & \xleftarrow{\alpha_1} & C_2 & \xleftarrow{\alpha_2} & \dots & \xleftarrow{\alpha_{n-1}} & C_n & \xleftarrow{\alpha_n} & C_0 \end{array}\right],$$

$$T_{-c}(\vec{\alpha}) = \left[\begin{array}{c|ccccccccc} & C_1 & \xleftarrow{\alpha_1} & C_2 & \xleftarrow{\alpha_2} & \dots & \xleftarrow{\alpha_{n-1}} & C_n & \longleftarrow & 0 \\ & \downarrow{\scriptstyle i_{C_1}} & & \downarrow{\scriptstyle i_{C_2}} & & & & \downarrow{\scriptstyle i_{C_n}} & & \downarrow \\ G(2)(g); & C_1 \oplus C_0 & \xleftarrow{\alpha_1\oplus 1} & C_2 \oplus C_0 & \xleftarrow{\alpha_2\oplus 1} & \dots & \xleftarrow{\alpha_{n-1}\oplus 1} & C_n \oplus C_0 & \xleftarrow{\alpha_n,1} & C_0 \\ & \downarrow{\scriptstyle \pi_{C_0}} & & \downarrow{\scriptstyle \pi_{C_0}} & & & & \downarrow{\scriptstyle \pi_{C_0}} & & \| \\ & C_0 & = & C_0 & = & \dots & = & C_0 & = & C_0 \end{array}\right],$$

$$T_t(\vec{\alpha}) = \left[\begin{array}{c|ccccccccc} & C_0 & = & C_0 & = & \dots & = & C_0 & = & C_0 \\ & \downarrow{\scriptstyle i_{C_0,\alpha_{1\cdots n}}} & & \downarrow{\scriptstyle 1_{C_0,\alpha_{2\cdots n}}} & & & & \downarrow{\scriptstyle i_{C_0,\alpha_n}} & & \| \\ G(3)(g); & C_0 \oplus C_1 & \xleftarrow{1\oplus\alpha_1} & C_0 \oplus C_2 & \xleftarrow{1\oplus\alpha_2} & \dots & \xleftarrow{1\oplus\alpha_{n-1}} & C_0 \oplus C_n & \xleftarrow{1,\alpha_n} & C_0 \\ & \downarrow{\scriptstyle \alpha_{1\cdots n}-1} & & \downarrow{\scriptstyle \alpha_{1\cdots n-1}-1} & & & & \downarrow{\scriptstyle \alpha_n-1} & & \downarrow \\ & C_1 & \xleftarrow{\alpha_1} & C_2 & \xleftarrow{\alpha_2} & \dots & \xleftarrow{\alpha_{n-1}} & C_n & \longleftarrow & 0 \end{array}\right].$$

The map $G(1)$ is the natural group homomorphism

$$G(C_1, C_0) \to G\begin{pmatrix} C_0 & 0 \\ \downarrow{\scriptstyle i_{C_0}} & \downarrow \\ C_0 \oplus C_1, & C_0 \\ \downarrow{\scriptstyle \pi_{C_1}} & \| \\ C_1 & C_0 \end{pmatrix}$$

given by the composite $G(\pi_{s_0(C_1)}, id) \circ G_{s_0}$, where $s_0 : S_1 \longrightarrow S_2$ is the degeneracy map taking C to $0 \to C = C$ and $\pi_{s_0(C_1)}$ is the projection map (of the direct sum)

$$\begin{pmatrix} C_0 \\ \downarrow{\scriptstyle i_{C_0}} \\ C_0 \oplus C_1 \\ \downarrow{\scriptstyle \pi_{C_1}} \\ C_1 \end{pmatrix} \to \begin{pmatrix} 0 \\ \downarrow \\ C_1 \\ \| \\ C_1 \end{pmatrix}.$$

The map $G(2)$ is the natural group homomorphism

$$G(C_1, C_0) \to G\left(\begin{array}{cc} C_1 & 0 \\ \downarrow{\scriptstyle i_{C_1}} & \downarrow \\ C_1 \oplus C_0, & C_0 \\ \downarrow{\scriptstyle \pi_{C_0}} & \| \\ C_0 & C_0 \end{array}\right)$$

given by the composite $(-1)G(\pi_{s_0(C_0)}, id) \circ G_{s_0} \circ G(\alpha_{1\cdots n}, id)$.

The map $G(3)$ is the difference of two natural maps $G_1(3)$ and $G_2(3)$

$$G(C_1, C_0) \to G\left(\begin{array}{cc} C_0 & C_0 \\ \downarrow{\scriptstyle i_{C_0},\alpha_{1\cdots n}} & \| \\ C_0 \oplus C_1, & C_0 \\ \downarrow{\scriptstyle \alpha_{1\cdots n}-1} & \downarrow \\ C_1 & 0 \end{array}\right).$$

The map $G_1(3)$ is given by the composite

$$G\left(\begin{array}{c} \alpha_{1\cdots n} \\ \pi_{C_1} \\ 0 \end{array}, id\right) \circ G_{s_1}$$

and the map $G_2(3)$ is given by the composite

$$G(\pi_{s_1(C_0)}) \circ G_{s_1} \circ G(\alpha_{1\cdots n}, id).$$

Now we note the following relations:

$$d_0 T_{id} = id, \qquad d_0 T_{-c} = -C, \qquad d_0 T_t = 0,$$

$$d_1 T_t = d_1 T_{id} + d_1 T_{-c}, \qquad d_2 T_{id} = d_2 T_{-c} = d_2 T_t = 0.$$

By additivity we obtain, in a $2N-1$ range,

$$\begin{aligned} id - C &= d_0 T_{id} + d_0 T_{-c} \\ &= (d_0 T_{id} + d_2 T_{id}) + (d_0 T_{-c} + d_2 T_{-c}) \\ &\simeq d_1 T_{id} + d_1 T_{-c} \\ &= d_1 T_t \\ &\simeq d_0 T_t + d_2 T_t \\ &= 0, \end{aligned}$$

and hence the result. □

3. Computation of the stabilized tensor products

In this section we compute the stabilization (in the sense of 0.1) of n-fold tensor product functors. Our second step in this calculation is a generalization of methods used in [5] for the special case when $n = 1$. If one is only interested in the rational case, this step can be greatly simplified by appealing to a multi-simplicial argument using complexes similar to Hochschild homology (for categories, as in [13])—which becomes very reminiscent of techniques in [8], section 4.

Let $M_1, \dots, M_n$ be fixed R–bimodules and let $G(M_1, \dots, M_n)$ (or just G when $M_1, \dots, M_n$ are clear) be the local coefficient system of section 2 given by

$$G(\mathcal{A})(A, A') = \bigotimes_{i=1}^{n} Hom_{\mathcal{S}_I \mathcal{A}}(\mathcal{S}_I(A), \mathcal{S}_{M_i}(A')).$$

For $\sigma \in \Sigma_n$ (where Σ_n is the group of permutations on n–letters) we let

$$\sigma_* : G(M_1, \dots, M_n) \to G(M_{\sigma(1)}, \dots, M_{\sigma(n)}) \tag{3.1}$$

be the evident natural transformation by rearranging the n–fold tensor product.

We will simply write F^n_* for $F_*(\mathcal{A}; G)$ in this section. Thus:

$$[p] \longrightarrow F^n_p \equiv \bigoplus_{\vec{A} \in N_p \mathcal{A}} \bigotimes_{i=1}^{n} Hom_{\mathcal{S}_I \mathcal{A}}(\mathcal{S}_I A_1, \mathcal{S}_{M_i} A_0),$$

$$\vec{A} = A_1 \longleftarrow \cdots \longleftarrow A_p \longleftarrow A_0,$$

with the face and degeneracy operators as defined before.

For $\sigma \in \Sigma_n$, we let $F^\sigma \mathcal{A}$ be the simplicial abelian group:

$$[p] \longrightarrow F^\sigma_p \equiv \bigoplus_{\vec{A} \in N_p \mathcal{A}^n} Hom_{\mathcal{S}_I \mathcal{A}}(\mathcal{S}_I A_{1,1}, \mathcal{S}_{M_1} A_{0,\sigma(1)})$$
$$\otimes \cdots \otimes Hom_{\mathcal{S}_I \mathcal{A}}(\mathcal{S}_I A_{1,n}, \mathcal{S}_{M_n} A_{0,\sigma(n)}),$$

$$\vec{A} = \{A_{1,i} \longleftarrow \cdots \longleftarrow A_{n,i} \longleftarrow A_{0,i}\}_{i=1}^n,$$

with face and degeneracy operators like those for F^n only being careful about order. We define

$$F^{\Sigma_n} = \bigoplus_{\sigma \in \Sigma_n} F^\sigma.$$

We define a map of simplicial abelian groups $\psi_\sigma : F^n \longrightarrow F^\sigma$ by taking the indexing nerve to the diagonal of the product and $\alpha \in \bigotimes_{i=1}^n Hom_{\mathcal{S}_I \mathcal{A}}(\mathcal{S}_I A_1, \mathcal{S}_{M_i} A_0)$ to itself. We define the natural transformation Δ from F^n to F^{Σ_n} by $\Delta = \bigoplus \psi_\sigma$. There is a natural Σ_n–action on F^n given by permuting the n–fold tensors. To make Δ an equivariant map we give F^{Σ_n} a Σ_n–action by:

(1) permuting the index set $N_p \mathcal{A}^n$,
(2) rearranging the n–fold tensors, and
(3) *conjugation* on the indexing element of Σ_n.

Thus, $(\alpha_1, \dots, \alpha_n; \sigma) * \tau = (\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)}; \tau^{-1}\sigma\tau)$.

So far, everything is well defined for an arbitrary linear category. Now we define $\phi_\sigma : F^\sigma \longrightarrow F^n$, which is a natural transformation of exact functors which preserve a chosen $\oplus$–action. We define ϕ_σ of $(\{A_{1,i} \overset{\beta_{1,i}}{\longleftarrow} \cdots \overset{\beta_{n,i}}{\longleftarrow} A_{0,i}\}_{i=1}^n; \gamma_1 \otimes \cdots \otimes \gamma_n)$ to be

$$\left(\bigoplus A_{1,i} \overset{\bigoplus \beta_{1,i}}{\longleftarrow} \cdots \overset{\bigoplus \beta_{n,i}}{\longleftarrow} \bigoplus A_{0,i}\ ; \tilde{\gamma}_1 \otimes \cdots \otimes \tilde{\gamma}_n \right)$$

where

$$\gamma_j \in Hom_{\mathcal{S}_I \mathcal{A}}(\mathcal{S}_I A_{1,j}, \mathcal{S}_{M_j} A_{0,\sigma(j)})$$

and

$$\tilde{\gamma}_j \in Hom_{\mathcal{S}_I \mathcal{A}} \left(\mathcal{S}_I \bigoplus A_{1,i}, \mathcal{S}_{M_j} \bigoplus A_{0,i} \right)$$

is the unique morphism (as a natural transformation for all γ_j) which is γ_j and zeros elsewhere (using the identification $\mathcal{S}_{M_j} \bigoplus A_{0,i} \cong \bigoplus S_{M_j} A_{0,i}$). We define $\phi : F^{\Sigma_n} \longrightarrow F^n$ by $\phi = \Sigma\phi_\sigma$.

Proposition (3.2). *The maps Δ and ϕ are homotopy inverses of each other.*

Proof. The proof of this proposition is in several steps. Our proof uses three sub–lemmas. In each sub-lemma we will be constructing semi–simplicial homotopies (they satisfy the simplicial homotopy identities with respect to the face maps, see [11], section 5). Given a semi-simplicial homotopy $\{h_i\}$, one can construct a chain homotopy H by setting $H_n = \sum_{i=0}^n (-1)^i h_i$.

Sub–lemma 1.

$$\psi_\sigma \circ \phi_\tau \simeq \begin{cases} id, & \text{if } \sigma = \tau, \\ 0, & \text{if } \sigma \neq \tau. \end{cases}$$

We first note that $\psi_\sigma \circ \phi_\tau$ of $(\{A_{1,i} \stackrel{\beta_{1,i}}{\longleftarrow} \cdots \stackrel{\beta_{n,i}}{\longleftarrow} A_{0,i}\}_{i=1}^n ; \gamma_1 \otimes \cdots \otimes \gamma_n)$ is

$$(\{\bigoplus A_{1,j} \stackrel{\oplus\beta_{1,j}}{\longleftarrow} \cdots \stackrel{\oplus\beta_{n,j}}{\longleftarrow} \bigoplus A_{0,j}\}_{i=1}^n \; ; \tilde{\gamma}_1 \otimes \cdots \otimes \tilde{\gamma}_n).$$

We construct a semi–simplicial homotopy h_t as follows. Let $\pi_u : \bigoplus_{i=1}^n A_{k,i} \longrightarrow A_{k,u}$ be the natural projection. We set h_t $(0 \leq t \leq n)$ of $(\{A_{1,i} \stackrel{\beta_{1,i}}{\longleftarrow} \cdots \stackrel{\beta_{n,i}}{\longleftarrow} A_{0,i}\}_{i=1}^n ; \gamma_1 \otimes \cdots \otimes \gamma_n)$ to be

$$\left(\left\{ A_{1,i} \stackrel{\beta_{1,i}}{\longleftarrow} \cdots \stackrel{\beta_{t-1,i}}{\longleftarrow} A_{t,i} \stackrel{\pi_i}{\longleftarrow} \bigoplus_{j=0}^n A_{t,j} \stackrel{\oplus\beta_{t,i}}{\longleftarrow} \cdots \stackrel{\oplus\beta_{n,i}}{\longleftarrow} \bigoplus_{j=0}^n A_{0,j} \right\}_{i=1}^n ; \hat{\gamma}_1 \otimes \cdots \otimes \hat{\gamma}_n \right),$$

where $\gamma_i \in Hom_{\mathcal{S}_I\mathcal{A}}(\mathcal{S}_I A_{1,i}, \mathcal{S}_{M_i} A_{0,\sigma(i)})$ and $\hat{\gamma}_i \in Hom_{\mathcal{S}_I\mathcal{A}}(\mathcal{S}_I A_{1,i}, \mathcal{S}_{M_i} \bigoplus A_{0,i})$ is the unique morphism which is γ_i and zeros elsewhere. One can check that this gives the desired homotopy. In particular, $d_{n+1}h_n$ is 0 if $\sigma \neq \tau$ and the identity if $\sigma = \tau$.

Let $\mathcal{M}_n$ be the set of *all* set maps from $\{1, \ldots, n\}$ to itself. For each $\lambda \in \mathcal{M}_n$, we define θ^λ, a simplicial self map of $F^n\mathcal{A}$, by sending $(A_1 \stackrel{\beta_1}{\longleftarrow} \cdots \stackrel{\beta_n}{\longleftarrow} A_0 \; ; \gamma_1 \otimes \cdots \otimes \gamma_n)$ to

$$(A_1^{\oplus n} \stackrel{\beta_1^{\oplus n}}{\longleftarrow} \cdots \stackrel{\beta_n^{\oplus n}}{\longleftarrow} A_0^{\oplus n} \; ; \tilde{\gamma}_{1,\lambda(1)} \otimes \cdots \otimes \tilde{\gamma}_{n,\lambda(n)}),$$

where $\tilde{\gamma}_{i,j} \in Hom_{\mathcal{S}_I\mathcal{A}}(\mathcal{S}_I A_1^{\oplus n}, \mathcal{S}_{M_i} A_0^{\oplus n})$ is γ_i in the (i, j) position and zeros elsewhere. We let $\Theta = \sum_{\lambda \in \mathcal{M}_n} \theta^\lambda$.

Sub–lemma 2. $\Theta \simeq id_{F^n}$.

We define the semi–simplicial homotopy $\{h_i\}$ as follows. Let $\delta : A \longrightarrow A^{\oplus n}$ be the diagonal map. We set h_i of $(A_1 \stackrel{\beta_1}{\longleftarrow} \cdots \stackrel{\beta_n}{\longleftarrow} A_0 \; ; \gamma_1 \otimes \cdots \otimes \gamma_n)$ to be

$$(A_1^{\oplus n} \stackrel{\beta_1^{\oplus n}}{\longleftarrow} \cdots \stackrel{\beta_i^{\oplus n}}{\longleftarrow} A_{i+1}^{\oplus n} \stackrel{\delta}{\longleftarrow} A_{i+1} \stackrel{\beta_{i+1}}{\longleftarrow} \cdots \stackrel{\beta_n}{\longleftarrow} A_0 \; ; \bar{\gamma}_1 \otimes \bar{\gamma}_2 \otimes \cdots \otimes \bar{\gamma}_n),$$

where $\bar{\gamma}_j \in Hom_{\mathcal{S}_I\mathcal{A}}(\mathcal{S}_I \bigoplus A_1, \mathcal{S}_{M_j} A_0)$ is $\gamma_j \circ \mathcal{S}_I \pi_j$, where π_j is the natural projection of $\bigoplus A_1$ onto its j-th coordinate. It is clear that $d_0 h_0$ is the identity. Similarly, $d_{n+1}h_n = \Theta$, since $d_{n+1}h_n$ is

$$\left((A_1^{\oplus n} \stackrel{\beta_1^{\oplus n}}{\longleftarrow} \cdots \stackrel{\beta_n^{\oplus n}}{\longleftarrow} A_0^{\oplus n} \; ; \sum_{j_1=1}^n \tilde{\gamma}_{1,j_1} \otimes \cdots \otimes \sum_{j_n=1}^n \tilde{\gamma}_{n,j_n} \right)$$

and by multilinearity

$$\sum_{j_1=1}^{n} \tilde{\gamma}_{1,j_1} \otimes \cdots \otimes \sum_{j_n=1}^{n} \tilde{\gamma}_{n,j_n} = \sum_{j_1,\dots,j_n=1}^{n} \tilde{\gamma}_{1,j_1} \otimes \cdots \otimes \tilde{\gamma}_{n,j_n}$$
$$= \sum_{\lambda \in \mathcal{M}_n} \tilde{\gamma}_{1,\lambda(1)} \otimes \cdots \otimes \tilde{\gamma}_{n,\lambda(n)}.$$

Sub–lemma 3. $\theta^\lambda \simeq 0$ *if* λ *is not surjective.*

For $\lambda \in \mathcal{M}_n$, we let $[A]_\lambda \in Hom_{\mathcal{A}}(A^{\oplus n}, A^{\oplus n})$ be the morphism which is the identity in positions $(i, \lambda(i))$ for all $1 \leq i \leq n$ and zeros elsewhere. Suppose λ is not surjective and let $k \notin \mathrm{Image}(\lambda)$. We define a semi-simplicial homotopy by setting h_i of $(A_1 \xleftarrow{\beta_1} \cdots \xleftarrow{\beta_n} A_0\, ; \gamma_1 \otimes \cdots \otimes \gamma_n)$ to be

$$(A_1^{\oplus n} \xleftarrow{\beta_1^{\oplus n}} \cdots \xleftarrow{\beta_i^{\oplus n}} A_{i+1}^{\oplus n} \xleftarrow{[A_{i+1}]_\lambda} A_{i+1}^{\oplus n} \xleftarrow{\beta_{i+1}^{\oplus n}} \cdots \xleftarrow{\beta_n^{\oplus n}} A_0^{\oplus n}\, ; \tilde{\gamma}_{1,1} \otimes \cdots \otimes \tilde{\gamma}_{n,n}).$$

Then $d_0 h_0 = \theta^\lambda$ and $d_{n+1} h_n = 0$, since $k \neq \{1, \dots, n\}$ implies that $[A_1]_\lambda \circ \tilde{\gamma}_{k,k} = 0$.

Proof of proposition 3.2. The composite map $\Delta \circ \phi$ is equal to $\Sigma \psi_\sigma \circ \phi_\tau$, which is homotopic to $\sum id_\sigma$ by sub–lemma 1. The composite map $\phi \circ \Delta$ is equal to $\sum \phi_\sigma \circ \psi_\sigma = \sum \theta^\sigma$, which by sub–lemma 3 is homotopic to Θ, which by sub–lemma 2 is homotopic to the identity. □

We now define a new construction $F^{\mathcal{S}_n}$. Let $\mathcal{S}_n$ be the *subset* of Σ_n consisting of cycles of length n. We note that $\mathcal{S}_n$ is an invariant subset by conjugation. Also, $|\mathcal{S}_n| = (n-1)!$, and if we let $\omega = (1\,2 \dots n)$ then $\mathcal{S}_n = \{\gamma^{-1}\omega\gamma | \gamma \in \Sigma_n\}$. We define $F^{\mathcal{S}_n}$ to be the subsimplicial abelian group of F^{Σ_n} determined by

$$F^{\mathcal{S}_n} = \bigoplus_{\sigma \in \mathcal{S}_n} F^\sigma.$$

We note that $F^{\mathcal{S}_n}$ is a Σ_n–invariant subsimplicial abelian group of F^{Σ_n}.

Lemma (3.3). *If* $\mathcal{A}_*$ *is an* n*–reduced simplicial exact category, then* $F^{\mathcal{S}_n}\mathcal{A}_* \longrightarrow F^{\Sigma_n}\mathcal{A}_*$ *is a* $2n-1$ *connected map.*

Proof. If σ can be written as the product of two disjoint cycles $\tau \circ \gamma$, then $F^\sigma \cong F^\tau \otimes F^\gamma$, and hence $F^\sigma \mathcal{A}_*$ would be at least $2n$ connected. Thus, $F^{\Sigma_n}\mathcal{A}_* = F^{\mathcal{S}_n}\mathcal{A}_* \oplus$ (terms $2n$ connected and higher). □

Theorem (3.4). *Using the notation of 0.1, if* $M_1 = M_2 = \cdots = M_n$ *then* $(F^n)^{st}$ *is naturally* Σ_n *equivalent to* $\mathbf{Z}[\Sigma_n] \otimes_{\mathbf{Z}[C_n]} (F^\omega)^{st}$.

Proof. By 3.2, $\Delta : F^n \longrightarrow F^{\Sigma_n}$ is a homotopy equivalence for all $\mathcal{A} \in \mathcal{S}_{\mathcal{P}}$, and hence by 3.3 we obtain

$$(F^n)^{st} = \lim_{k \to \infty} F^n S^k[-k] \xrightarrow{\simeq} \lim_{k \to \infty} F^{\Sigma_n} S^k[-k] \xleftarrow{\simeq} \lim_{k \to \infty} F^{\mathcal{S}_n} S^k[-k].$$

Now we note that the natural map $\mathbf{Z}[\Sigma_n] \otimes_{\mathbf{Z}[C_n]} F^\omega \longrightarrow F^{\mathcal{S}_n}$ given by sending $(\sigma \otimes x)$ to $\sigma_*(x)$ (as defined in 3.1) is a Σ_n equivariant isomorphism of simplicial abelian groups. □

We now slightly generalize theorem 1.4 of [5] so that we may rewrite $(F^\omega)^{st}$.

Definition. Given a linear functor G from $\mathcal{A}$ to $\mathcal{B}$, we define the "twisted" product category $\mathcal{A}_G\mathcal{B}$ as follows. We set $\mathrm{Obj}(\mathcal{A}_G\mathcal{B})$ to be $\mathrm{Obj}(\mathcal{A}) \times \mathrm{Obj}(\mathcal{B})$ and

$$\mathrm{Hom}_{\mathcal{A}_G\mathcal{B}}((A,B),(A',B')) = \mathrm{Hom}_{\mathcal{A}}(A,A') \oplus \mathrm{Hom}_{\mathcal{B}}(B,B') \oplus \mathrm{Hom}_{\mathcal{B}}(G(A),B')$$

with composition defined by $(f,g,h)\circ(f',g',h') = (f\circ f', g\circ g', h\circ G(f') + g\circ h')$.

For fixed $X = (A,B), Y = (A',B') \in \mathcal{A}_G\mathcal{B}$, let C_* be the simplicial abelian group

$$[p] \longrightarrow C_p \equiv \bigoplus_{\vec{C}\in N_p\mathcal{A}_G\mathcal{B}} Hom_{\mathcal{A}_G\mathcal{B}}(C_1, X) \otimes Hom_{\mathcal{A}_G\mathcal{B}}(Y, C_0),$$
$$\vec{C} = C_1 \longleftarrow \cdots \longleftarrow C_p \longleftarrow C_0,$$

with the face and degeneracies given like those for F^n. We will represent an arbitrary generating element of C_n by $(\beta_0 \otimes \alpha_0; \alpha_1, \ldots, \alpha_n)$. We let $F^{(1)}(\mathcal{A})$ be the simplicial functor

$$[p] \mapsto F_p^{(1)}(\mathcal{A}) = \bigoplus_{\vec{A}\in N_p\mathcal{A}} Hom_{\mathcal{A}}(A_1, A) \otimes Hom_{\mathcal{A}}(A', A_0),$$
$$\vec{A} = A_1 \longleftarrow \cdots \longleftarrow A_p \longleftarrow A_0,$$

and let $F^{(1)}(\mathcal{B})$ be the simplicial functor

$$[p] \mapsto F_p^{(1)}(\mathcal{B}) = \bigoplus_{\vec{B}\in N_p\mathcal{B}} Hom_{\mathcal{B}}(B_1, B) \otimes Hom_{\mathcal{B}}(B', B_0),$$
$$\vec{B} = B_1 \longleftarrow \cdots \longleftarrow B_p \longleftarrow B_0,$$

with the face and degeneracies given like those for F^n.

Proposition (3.5). *The functor from $\mathcal{A}_G\mathcal{B}$ to $\mathcal{A}\times\mathcal{B}$ which is the identity on objects (and sends (f,g,h) to $f\times g$) produces a homotopy equivalence from C_* to $F^{(1)}(\mathcal{A}) \times F^{(1)}(\mathcal{B})$.*

Proof. We will (once again) be defining several chain homotopies which arise from semi–simplicial homotopies (they satisfy the simplicial homotopy identities with respect to the face maps; see [11], section 5). Given a semi-simplicial homotopy $\{h_i\}$, one can construct a chain homotopy H by setting $H_n = \sum_{i=0}^n (-1)^i h_i$. □

<u>First reduction</u>: The subcomplex of C_* generated by elements of the form $(\beta_0 \otimes \alpha_0; \alpha_1, \ldots, \alpha_n)$ such that $\alpha_0 = (0,0,h_0)$ and $\beta_0 = (0,0,h_0')$ is acyclic.

<u>*Proof*</u>. We let $\alpha_i = (f_i, g_i, h_i)$ and define a semi-simplicial homotopy from the identity to 0 as follows:

$$h_i((0,0,h_0') \otimes (0,0,h_0); \alpha_1, \ldots, \alpha_n)$$
$$= ((0,0,h_0') \otimes (0,0,h_0); (0,g_1,0), \ldots, (0,g_i,0), (0, id_{B_{i+1}}, 0), \alpha_{i+1}, \ldots, \alpha_n).$$

Now we quotient C_* by this acyclic subcomplex to get a new complex $\tilde{C}_*$. We will write a generating element of this complex as $((f_0', g_0', \star') \otimes (f_0, g_0, \star); \alpha_1, \dots, \alpha_n)$. The complex $\tilde{C}_*$ splits into a sum of four subcomplexes: $\tilde{C}_* = AA_* \oplus AB_* \oplus BA_* \oplus BB_*$, where

$$\begin{aligned} AA_n &\text{ is generated by } ((f_0', 0, \star') \otimes (f_0, 0, \star); \alpha_1, \dots, \alpha_n), \\ AB_n &\text{ is generated by } ((f_0', 0, \star') \otimes (0, g_0, \star); \alpha_1, \dots, \alpha_n), \\ BA_n &\text{ is generated by } ((0, g_0', \star') \otimes (f_0, 0, \star); \alpha_1, \dots, \alpha_n), \\ BB_n &\text{ is generated by } ((0, g_0', \star') \otimes (0, g_0, \star); \alpha_1, \dots, \alpha_n). \end{aligned}$$

□

Second reduction: The projection from $AA_* \oplus AB_*$ to $F^{(1)}\mathcal{A}$ generated by sending $((f_0', 0, \star') \otimes (f_0, 0, \star); \alpha_1, \dots, \alpha_n)$ to $(f_0' \otimes f_0; f_1, \dots, f_n)$ and AB_* to 0 is a homotopy equivalence.

Proof. Choose elements a of $\mathcal{A}$ and b of $\mathcal{B}$. The projection has a section defined by sending $(f_0' \otimes f_0; f_1, \dots, f_n)$ to the equivalence class containing

$$((f_0', 0_b, \star) \otimes (f_0, 0_b, \star); (f_1, 0_b, 0), \dots, (f_n, 0_b, 0)),$$

where we let 0_b denote the zero endomorphism of b. A simplicial homotopy from the identity to the composite can be defined by sending the class of

$$((f_0', 0, \star') \otimes (f_0, 0, \star); \alpha_1, \dots, \alpha_n) \oplus ((\hat{f}_0', 0, \star') \otimes (0, \hat{g}_0, \star); \hat{\alpha}_1, \dots, \hat{\alpha}_n)$$

by h_i to the class of

$$\begin{gathered} ((f_0', 0, \star') \otimes (f_0, 0, \star); (f_1, 0_b, 0), \dots, (f_i, 0_b, 0), (id_{A_{i+1}}, 0, 0), \alpha_{i+1}, \dots, \alpha_n) \\ \oplus \\ ((\hat{f}_0', 0, \star') \otimes (0, \hat{g}_0, \star); (0_a, \hat{g}_1, 0), \dots, (0_a, \hat{g}_i, 0), (0, id_{\hat{B}_{i+1}}, 0), \hat{\alpha}_{i+1}, \dots, \hat{\alpha}_n). \end{gathered}$$

□

Third reduction: The projection from $BB_* \oplus BA_*$ to $F^{(1)}\mathcal{B}$ generated by sending a generating element $((0, g_0', \star') \otimes (0, g_0, \star); \alpha_1, \dots, \alpha_n)$ to $(g_0' \otimes g_0; g_1, \dots, g_n)$ and BA_* to 0 is a homotopy equivalence.

Proof. Choose some element a of $\mathcal{A}$. The projection has a section defined by sending $(g_0; g_1, \dots, g_n)$ to the equivalence class containing

$$((0_a, g_0', \star') \otimes (0_a, g_0, \star); (0_a, g_1, 0), \dots, (0_a, g_n, 0)),$$

where we let 0_a denote the zero endomorphism of a. A simplicial homotopy from the composite to the identity can be defined by sending the class of

$$((0, g_0', \star') \otimes (0, g_0, \star); \alpha_1, \dots, \alpha_n) \oplus (0, \hat{g}_0', \star') \otimes (\hat{f}_0, 0, \star); \hat{\alpha}_1, \dots, \hat{\alpha}_n)$$

by h_i to the class of

$$\begin{gathered} ((0, g_0', \star') \otimes (0, g_0, \star); \alpha_1, \dots, \alpha_i, (0, id_{B_{i+1}}, 0), (0_a, g_{i+1}, 0), \dots, (0_a, g_n, 0)) \\ \oplus \\ (0, \hat{g}_0', \star') \otimes (\hat{f}_0, 0, \star); \hat{\alpha}_1, \dots, \hat{\alpha}_i, (0, id_{\hat{B}_{i+1}}, 0), (0_a, \hat{g}_{i+1}, 0), \dots, (0_a, \hat{g}_n, 0)). \end{gathered}$$

We have constructed a diagram of complexes

$$\begin{array}{ccc} F^{(1)}\mathcal{A} \times F^{(1)}\mathcal{B} & \xrightarrow{inc} & C_* \\ \uparrow \simeq & & \downarrow \simeq \\ AA_* \oplus AB_* \oplus BA_* \oplus BB_* & \xleftarrow{\cong} & \tilde{C}_* \end{array}$$

with the maps up and down quasi–isomorphisms by reductions 1–3 above. Since the composite around the square is the identity on $F^{(1)}\mathcal{A} \times F^{(1)}\mathcal{B}$, we see that *inc* is a quasi–isomorphism. Since the inclusion *inc* is a section to our map, we are done. □

Corollary (3.6). *For $f : \mathcal{A} \to \mathcal{B}$ a linear functor, the natural map*

$$F^\omega(\mathcal{A}) \oplus F^\omega(\mathcal{B}) \to F^\omega(\mathcal{A}_f\mathcal{B})$$

is an equivalence.

Proof. We can consider F^ω as the diagonal of an n–simplicial abelian group. By the Eilenberg-Zilber theorem it suffices to show that the map is an equivalence on the associate n–dimensional complexes. We can factor the map into n-steps, where we pass from the product category to the twisted category in each simplicial dimension one at a time separately. Each of these maps is levelwise an example of proposition 3.5 except for the tensor of a module. However, since proposition 3.5 was obtained by chain homotopies, it remains true after tensoring with a fixed module. Thus, each of the n maps is an equivalence by the realization lemma (or standard spectral sequence arguments), and we are finished. □

Corollary (3.7). *For any n we have $\Omega F^\omega S \xrightarrow{\simeq} (F^\omega)^{st}$, and if the exact category $\mathcal{C}$ is split (all cofibrations have a retract), then $F^\omega\mathcal{C} \xrightarrow{\simeq} \Omega F^\omega S.\mathcal{C}$.*

The proof is exactly as in section 1 of [5]).

Corollary (3.8). *If $M_1 = M_2 = \cdots = M_n$, then the functor $(F^n)^{st}$ is naturally Σ_n equivalent (in the weak sense—that is, connected by a chain of Σ_n–equivariant maps which are also equivalences non-equivariantly) to $\Omega[\mathbf{Z}[\Sigma_n] \otimes_{\mathbf{Z}[C_n]} (F^\omega S.\ )]$, and in particular*

$$(F^n)^{st}(\mathcal{P}) \sim_{\Sigma_n} \mathbf{Z}[\Sigma_n] \otimes_{\mathbf{Z}[C_n]} F^\omega(\mathcal{P}).$$

We will now rewrite our functors F^ω in terms of Hochschild homology when $\mathbf{Q} \subseteq R$. More generally, one should use Mac Lane homology to rewrite these, which we will do in the next section.

There is a natural simplicial map τ from $F^\omega(\mathcal{P})$ to $HH(R^{\otimes n}, (\bigotimes_{j=1}^n M_i)_\tau)$ which is $\mathbf{Z}[C_n]$–equivariant when the M_i's are equal given in simplicial dimension $p-1$ by the composite shown in Figure 1, which is given by the evaluation maps $\mathbf{Z}[G] \to G$ which take $\sum z_i[g_i]$ to $\sum z_i \cdot g_i$ for any abelian group G, and where the map "trace" is the Dennis trace map (as used in [13]).

Proposition (3.9). *If $\mathbf{Q} \subseteq R$, then the simplicial map τ from $F^\omega\mathcal{P}$ to*

$$HH\left(R^{\otimes n}, \left(\bigotimes_{j=1}^{n} M_i\right)_\tau\right),$$

is an equivalence which is C_n–equivariant when the M_i's are equal.

$$F^{\omega}_{p-1}(\mathcal{P}) = \bigoplus_{A_0,\dots,A_{np-1}\in\mathcal{P}}$$

$$\left(\begin{array}{cccc} Hom_R(A_1, A_0 \otimes_R M_1) & \otimes & \mathbf{Z}[Hom_R(A_2, A_1)] & \otimes \cdots \otimes \mathbf{Z}[Hom_R(A_{p-1}, A_p)] \\ Hom_R(A_{p+1}, A_p \otimes_R M_2) & \otimes & \mathbf{Z}[Hom_R(A_{p+2}, A_{p+1})] & \otimes \cdots \otimes \mathbf{Z}[Hom_R(A_{2p-1}, A_{2p})] \\ \vdots & & \vdots & \vdots \\ Hom_R(A_{(n-1)p+1}, A_{(n-1)p} \otimes_R M_n) & \otimes & \mathbf{Z}[Hom_R(A_{(n-1)p+2}, A_{(n-1)p+1})] & \otimes \cdots \otimes \mathbf{Z}[Hom_R(A_{np-1}, A_0)] \end{array}\right)$$

$$\Big\downarrow \text{evaluation}$$

$$\bigoplus_{A_0,\dots,A_{np-1}\in\mathcal{P}}$$

$$\left(\begin{array}{cccc} Hom_R(A_1, A_0 \otimes_R M_1) & \otimes & Hom_R(A_2, A_1) & \otimes \cdots \otimes Hom_R(A_{p-1}, A_p) \\ Hom_R(A_{p+1}, A_p \otimes_R M_2) & \otimes & Hom_R(A_{p+2}, A_{p+1}) & \otimes \cdots \otimes Hom_R(A_{2p-1}, A_{2p}) \\ \vdots & & \vdots & \vdots \\ Hom_R(A_{(n-1)p+1}, A_{(n-1)p} \otimes_R M_n) & \otimes & Hom_R(A_{(n-1)p+2}, A_{(n-1)p+1}) & \otimes \cdots \otimes Hom_R(A_{np-1}, A_0) \end{array}\right)$$

$$\Big\downarrow \text{trace}$$

$$\begin{pmatrix} M_1 \\ \otimes \\ M_2 \\ \otimes \\ \vdots \\ \otimes \\ M_n \end{pmatrix} \otimes \begin{pmatrix} R \\ \otimes \\ R \\ \otimes \\ \vdots \\ \otimes \\ R \end{pmatrix} \otimes \begin{pmatrix} R \\ \otimes \\ R \\ \otimes \\ \vdots \\ \otimes \\ R \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} R \\ \otimes \\ R \\ \otimes \\ \vdots \\ \otimes \\ R \end{pmatrix}$$

FIGURE 1.

Proof. The map τ is the diagonal of an n–dimensional simplicial map we call τ'. By the Eilenberg-Zilber theorem, it suffices to show τ' is an equivalence. The map τ' can be decomposed as the composite of n maps of n-simplicial abelian groups, where one applies the map "τ" to one dimension at a time. Levelwise, each of these maps is a rational equivalence by 1.4.3 and 1.4.8 of [7], and hence by the realization lemma (or a standard spectral sequence argument) each of the n-composites is a rational equivalence, and so τ' is a rational equivalence. □

4. Relation to Mac Lane homology

In this section we rewrite our computation from section 3 in terms of Mac Lane homology. We will assume the reader is familiar with Mac Lane's Q–construction (e.g. [9] or [10]).

We can rewrite F_* as

$$F_n(\mathcal{C};D) = \bigoplus_{A_0,\dots,A_n\in\mathcal{C}} D(A_1,A_0)\otimes_{\mathbf{Z}} \mathbf{Z}[Hom_{\mathcal{C}}(A_2,A_1)] \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} \mathbf{Z}[Hom_{\mathcal{C}}(A_0,A_n)],$$

where $\mathbf{Z}[X]$ is the free abelian group generated by a pointed set X.

For any abelian group, we have natural maps $\mathbf{Z}[G] \xrightarrow{\beta} Q_*(G) \xrightarrow{\gamma} G$, where $Q_*(G)$ is Mac Lane's Q–construction (an explicit chain construction whose homology is the stable homology of the group G). We also recall that there is a natural map $Q_*(G)\otimes_{\mathbf{Z}} Q_*(G') \xrightarrow{\mu} Q_*(G\otimes_{\mathbf{Z}} G')$ which can be used to give $Q_*(R)$ the structure of a differential graded algebra when R is a ring and such that γ_R becomes a map of differential graded algebras. We note that the natural map $Z[G]\otimes Z[G'] \cong Z[G\times G'] \to Z[G\otimes_{\mathbf{Z}} G']$ commutes with μ via β.

Let D be a *bi-additive* functor. Set $Q_*(\mathcal{C};D)$ to be the simplicial chain complex defined by

$$Q_n(\mathcal{C};D) = \bigoplus_{A_0,\dots,A_n\in\mathcal{C}} D(A_1,A_0)\otimes_{\mathbf{Z}} Q_*(Hom_{\mathcal{C}}(A_2,A_1)) \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} Q_*(Hom_{\mathcal{C}}(A_0,A_n))$$

with simplicial structure maps like those of F_* using the natural maps γ for d_0 and d_{n+1} and μ otherwise. By 1.4.8 of [7], $Q_*(\mathcal{C};D)$ is naturally equivalent to $TH(\mathcal{C};D)$. Using β, we obtain a natural transformation of simplicial objects

$$(*) \qquad F_*(\mathcal{C};D) \xrightarrow{\beta} Q_*(\mathcal{C};D),$$

which is an equivalence for $\mathcal{C}$ a split exact category by the main result of [16]. We note that one can also obtain this result by modifying the proof of proposition 2.1 using $F_0(\mathcal{C};D) = Q_0(\mathcal{C};D)$.

We identify R with the subcategory of $\mathcal{P}$ generated by a free module of rank 1. By inclusion of subcategories, we obtain a map of simplicial objects

$$(**) \qquad Q_*(R;D|_R) \xrightarrow{i} Q_*(\mathcal{P};D),$$

which is a natural equivalence by 2.1.5 of [7].

Theorem (4.1). *Let M be an R–bimodule and let $G(M, \dots, M)$ be the the local coefficient system on $\mathcal{P}$ as defined in section 2. Then*

$$G(M, \dots, M)^{st}(\mathcal{P}) \sim_{\Sigma_n} \mathbf{Z}[\Sigma_n] \otimes_{\mathbf{Z}[C_n]} HH\left(Q_*(R)^{\otimes n}; M_\tau^{\otimes n}\right),$$

where HH is the Hochschild homology complex for $Q_*(R)^{\otimes n}$ acting on the bimodule by

$$(m_1 \otimes \cdots \otimes m_n) * (q_1 \otimes \cdots \otimes q_n) = (m_1 q_1 \otimes m_2 q_2 \otimes \cdots \otimes m_n q_n),$$

$$(q_1 \otimes \cdots \otimes q_n) * (m_1 \otimes \cdots \otimes m_n) = (q_n m_1 \otimes q_1 m_2 \otimes \cdots \otimes q_{n-1} m_n),$$

and C_n is the cyclic group of n elements, which acts by cyclic permutations—the equivalence is weakly Σ_n–equivariant.

Proof. By corollary 3.8, it suffices to show that F^ω is weakly equivalent to

$$HH(Q_*(R)^{\otimes n}; M_1 \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} M_n)$$

in a C_n-equivariant manner when the M_i's are equal. Just as we defined Q_* for F_*, we can define Q_*^ω with (appropriately C_n equivariant) maps

$$F^\omega(\mathcal{P}) \xrightarrow{\beta'} Q^\omega(\mathcal{P}) \xleftarrow{i'} Q^\omega(R).$$

Since $Q^\omega(R)$ is isomorphic to $HH(Q_*(R)^{\otimes n}; M_1 \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} M_n)$, we simply need to show that both β' and i' are equivalences. However, both these maps are the diagonal maps of n–fold multi-simplicial maps, and these n–fold multi-simplicial maps are n–fold composite maps, each of which is an equivalence by repeated applications of (*) and (**) above. □

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